The Fool's Errand Adventure

A Sightseeing Tour for Intuition

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The Fool's Errand Adventure

To my younger self – Some intuition I wish I'd found earlier along the path…

The quest has become an adventure where the questions are more important than the answers. It seems to be a tale worth sharing…

The challenge begins with a nagging desire to find the meaning of, and some intuition about, "the most famous of all formulas,"[i] Euler's identity:

$$e^{i\pi} + 1 = 0 \qquad \text{eq 1}$$

The equation, "surely … among the most beautiful formulas in all of mathematics," is valued for uniting the special mathematical constants *"e," "i"* and *"π"* with the important "identity numbers" of "0" and "1" and with the fundamental operations: addition, multiplication and exponent-powers. But what does it mean? Is there any intuition to be gained from better understanding it?

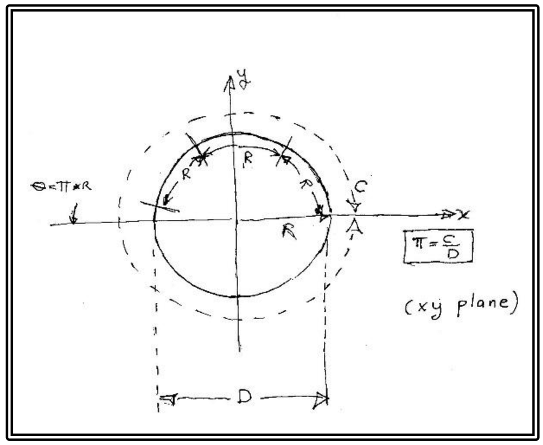
In just a bit, our adventure will begin with a detour as we consider the **special constant "*i* "** which represents $\sqrt{-1}$ and is associated with a unit vector (length =1) perpendicular (∟) to the real number line. Although called an "imaginary" number, **"*i* "** has wonderful properties and is used extensively in engineering. However, for purposes of our adventure, Euler's equation can be rewritten in more explicit form by replacing **"*i* "** with $\sqrt{-1}$:

$$e^{\pi * \sqrt{-1}} + 1 = 0 \qquad \text{eq 2}$$

Those more comfortable with the more usual complex plane usage, ***"i,"*** may read $\sqrt{-1}$ as such.

The **special constant "π"** is familiar and associated with circles. It is an irrational number a bit larger than 3 and is defined as the ratio of the circumference of a circle as compared to its diameter or the inside length of a straight line through the circle center. For our purposes it is helpful to note that there are π "radians" in the circumference of a half circle, radians being arc-lengths along the edge of circle measured in units equaling the length of the radius of the circle. Figure 1 shows these relationships plotted in the familiar x-y plane.

FIGURE 1



XY Plane

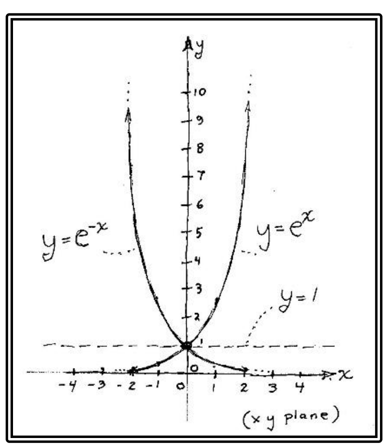
Meanwhile, **special constant "e",** an irrational number about two thirds of the way between 2 and 3, also has wonderful properties. It is used as the typical base for the "exponential functions" of the form:

$$y = e^x \qquad \text{eq 3}$$

Exponential functions are used when the growth or decay of some "thing" is directly related to current amount of the "thing." For example, water in a bucket might leak quickly when it is full but leak more slowly as the level drops. A biological example is that a small population grows slowly, while a large population of the same species grows explosively. Figure 2 shows the XY plot of classic exponential growth function, $y = e^x$, and the related decay function, $y = e^{-x}$.

The closely related exponential-type equations ($y = 2^x$) and ($y = \pi^x$) give plots remarkably similar to the curve shown in Figure 2. Figure 2 also shows the XY plot of the function y = f(x) = 1, a dashed horizontal line through the y=1 point on the y- axis.

FIGURE 2



XY Plane

What makes base *"e"* and the classic exponential function so special is the unique property that the derivative or slope of the function at each point is equal to the value of the function itself. The equation is its own derivative. This is not true for the other exponential-type equations with bases of 2, π or any other numbers.

PERSPECTIVES

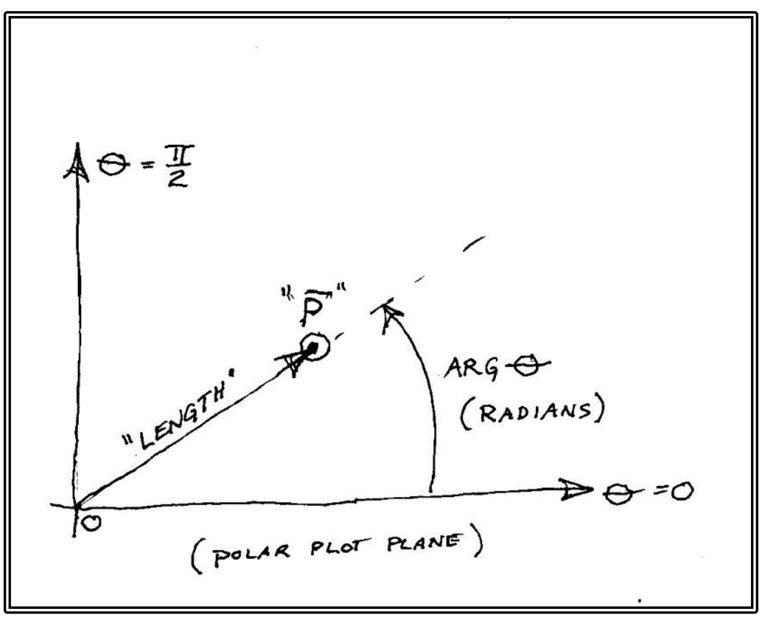
The Adventure will involve looking at functions from a variety of perspectives or points of view. The XY plots, shown above, are a familiar perspective. Find your "x" input (independent variable") along the horizontal X axis and see the function value, Y(x), plotted as a height directly above the x point and measured against the vertical Y axis which is perpendicular (∟) to the X axis.

A "Polar-Plot-Plane" offers an alternative perspective based on a vector plane – mapping (~») distance and direction from the origin (0 point) to a point **P**. The point **P** can be specified by an ordered pair of the form:

$$\mathbf{P} \sim\!\!\gg \{\, |\mathit{distance}|\, ,\ \mathit{argument} \}\,, \qquad \text{eq 4}$$

where **|distance|** is the absolute magnitude of "length" of the vector from 0 to point ***P*** and **argument** is the angle measured in radians from the $(\theta = 0)$ axis, which is usually the horizontal axis. Traditionally, positive arguments are measured in a counterclockwise direction and negative arguments are measured in a clockwise direction. This is shown in Figure 3.

FIGURE 3



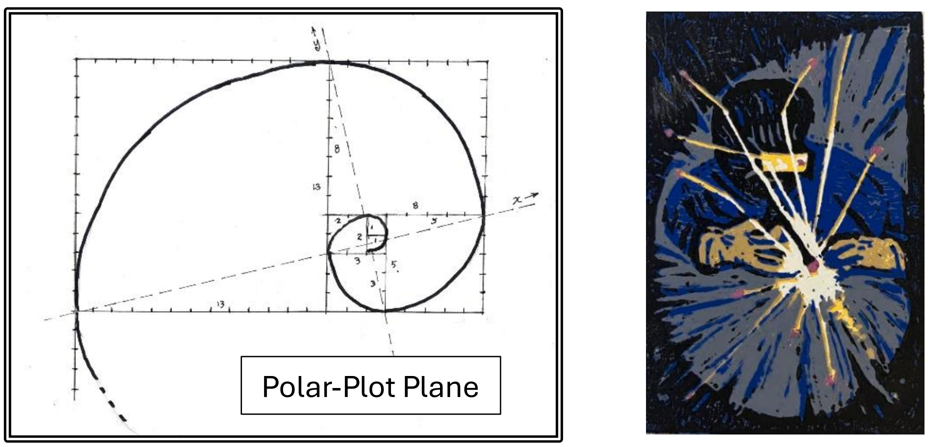
Re-plotting the function $y = e^x$ (eq. 3) from the XY-Plane in Figure 2 into a Polar-Plot-Plane equation (using angle Θ as the independent variable instead of “x”):

$$\boldsymbol{R}(\theta) = e^{\theta} \sim\!\text{»} \{ \,|e^{\theta}|\, , \theta \} \qquad \text{eq 5}$$

produces a startling change; the curve that sweeps off into the distance in Figure 2 becomes a growing spiral similar to the logarithmic “Golden Spiral” sketched in Figure 4:

Here we switch notation from “x” to “θ” and use curly-cue brackets, “ { } ,” to be clear about the polar angular nature of the independent input variable, θ, and to remind us that it is wrapping around the origin.

FIGURE 4A and 4B



Note that Figure 4A (left) shows a <u>*scaled version*</u> *of the logarithmic-spiral. Figure 4A actually plots the function* $\mathbf{P}(\theta) \sim e^{\,0.278*\theta} \sim (1.325)^{\,\theta}$. *The scaling changes how quickly the spiral grows. See Endnote[ii]. This specially scaled figure presents the Golden Spiral curve as often used by artists and nature literature. Also note that the figure shows the sprial fit to the inscribed Fibonacci sequence of squares. Note the slightly shifted x and y axes. Figure 4B (right) shows a block print image blatantly composed using a golden spiral.*

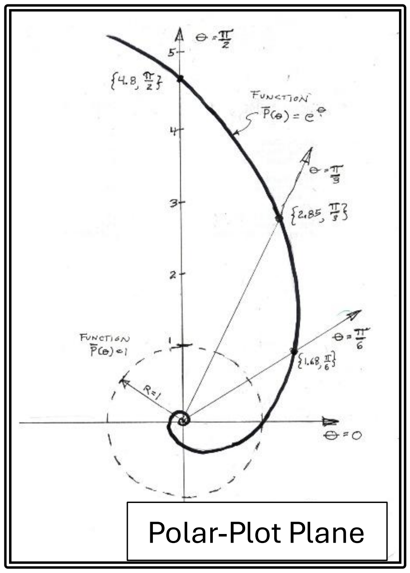
WHY do this replotting? For certain functions related to the exponential function this re-plot from the familiar XY-plane into the Polar-Plot-plane will be very handy later when multiplying vectors and finding roots and powers of vectors.

Meanwhile, this logarithmic “Golden Spiral” of Figure 4 fascinated the Greeks and classical artists and has its own interesting properties including inscribing an expanding series of “golden rectangles” [iii] and mimicking the Fibonacci sequence [iv].

Figure 5 zooms in toward the center of the logarithmic-spiral, replotting Eq. 5. $\mathbf{P}(\theta) = e^{\theta} \sim\!\text{»} \{ \,|e^{\theta}|\, , \theta \}$, this time without the scaling of Figure 4. A very interesting thing to note in

Figure 5 is that the dashed horizontal line of the function Y(x)=1 from Figure 2 now re-plots in the Polar Plot-Plane as $\boldsymbol{P}(\theta) = 1$~» { 1 , θ } and appears as a circle of radius (R=1) ("unit circle"), centered around the origin 0. The adventure will discover more about this unit circle.

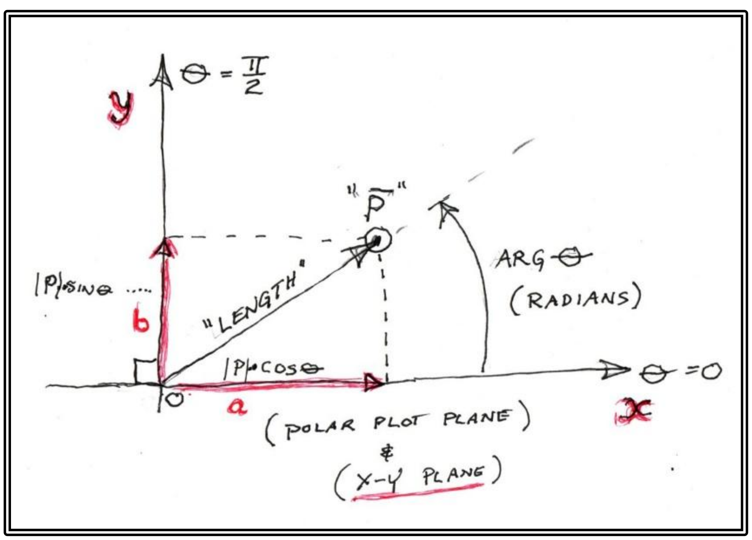
FIGURE 5



While in the Polar-Plot-plane, any point $\boldsymbol{P}(\theta)$ can also be described by a new rectangular set of coordinates ***[a, ⌊b],*** which are a (tail-to-head) sum of a vector ***a***, the projection of $\boldsymbol{P}(\theta)$ onto the $(\theta = 0)$ axis (horizontal) and a vector **⌊*b***, the projection of P onto the (θ =π/2) axis which is perpendicular (**⌊**) to the $(\theta = 0)$ axis. This is shown in Figure 6.

$\boldsymbol{P}(\theta)$~» ***[a, ⌊b].*** eq 6

FIGURE 6

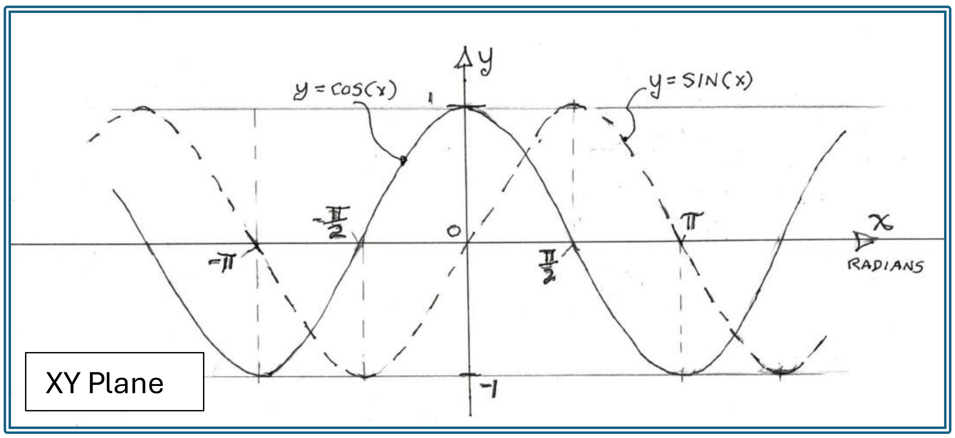


And one more, sometimes convenient, notation for this same point in the Polar-Plot-plane:

$$\boldsymbol{P}(\theta) \sim » (|\boldsymbol{P}| * cos(\theta)) + \llcorner * (|\boldsymbol{P}| * sin(\theta)) \qquad \text{eq 7}$$

Here the ***sin*** and ***cos*** trigonometric functions do the explicit work of projecting $\boldsymbol{P}(\theta)$ on the horizontal and vertical axes. But it must be remembered that these rectangular and vector plots of the function in the Polar-Plot-plane appear very differently in the XY plane as shown in the familiar mapping of the XY-plane sinusoids in Figure 7.

FIGURE 7



INSIGHT FROM ONE PERSPECTIVE TO ANOTHER:

Euler was playing with the classic exponential function ALGEBRAICALLY in its infinite power series form, the "Taylor Series" form. [v] The Taylor series was approximating the function from its slopes (or derivatives) in the XY-plane (see Fig 2):

$$e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \, , \; -\infty < x < \infty \qquad \text{eq 8}$$

As an example of the uniqueness of the exponential function (eq 8): if you use the ordinary rules to differentiate the 4th term, ($\frac{x^3}{3!}$) it becomes the (original) 3rd term, ($\frac{x^2}{2!}$), etc. and something similar happens for each term all the way to infinity… and beyond. For this reason, the exponential ends up being its own derivative (and its own second and third and additional derivatives, too!) This is unique.

Playing further with this, Euler substituted in "(ix)" into this series:

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots, \quad -\pi < x < +\pi \qquad \text{eq 9}$$

There is an important transition here. If ***"i"*** were a simple (real) constant like, say "2" or "π" this analysis could continue in the XY-plane.

HOWEVER, Euler was interpreting $i = \sqrt{-1}$, borrowing from the "complex plane", so every $(i)^2$ becomes (-1), and in the end "i" is only involved in the odd-powered terms; the equation becomes ...

$$e^{ix} = 1 + i\frac{x}{1!} - \frac{(x)^2}{2!} - i\frac{(x)^3}{3!} + \frac{(x)^4}{4!} + i\frac{(x)^5}{5!} - \frac{(x)^6}{6!} - i\frac{(x)^7}{7!} + \frac{(x)^8}{8!} + i\frac{(x)^9}{9!} \ldots,$$

eq 10

Euler took this complex number interpretation and separated out all the terms which involved the factor "i" and got a two-line sum:

$$e^{ix} = 1 - \frac{(x)^2}{2!} + \frac{(x)^4}{4!} - \frac{(x)^6}{6!} + \frac{(x)^8}{8!} + \cdots,$$

$$+i\frac{(x)}{1!} - i\frac{(x)^3}{3!} + i\frac{(x)^5}{5!} - i\frac{(x)^7}{7!} + i\frac{(x)^9}{9!} \ldots,$$

eq 11

Euler recognized the first line as the Taylor Series for the cosine function shown in Figure 7. This Taylor Series representation of the cosine function was well known in his day. The second line he recognized as the "i" multiplied by the well-known Taylor Series for the sine function [vi]. Figure 7, above, showed those equations plotted in the XY plane.

This led to Euler's famous equation which plots to (~») a unit circle (diameter = 1) in the Complex-Plane. Along the way, this adventure will note parallels to the Complex-Plane, but will generally avoid **using** complex numbers; the intent is to emphasize and build intuition. Figure 8 shows how e^{ix} maps into the unit circle.

$$e^{ix} = \cos x + i * \sin x \qquad \text{eq 12}$$

Solve this equation (12) at x = π, and it simplifies to:

$$e^{i\pi} = \cos\pi + i * \sin\pi \;\sim\!\!\gg\; [-1 + 0i] = -1$$

And rewriting, this becomes:

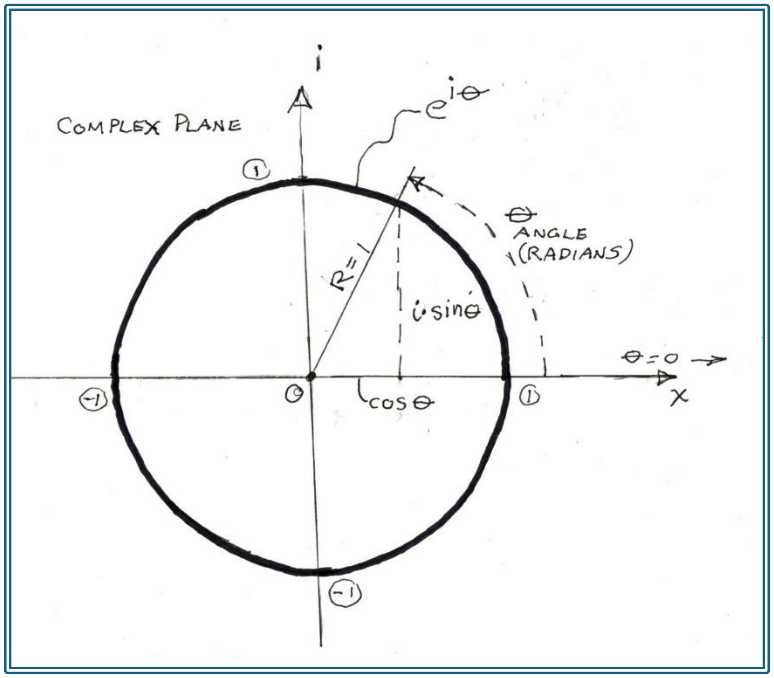
$$e^{i\pi} + 1 = 0 \qquad \text{eq 1 (repeated)}$$

Euler's famous identity, eq. 1, is a special case of the more general eq. 12, as a mapping in the Complex-Plane.

This Complex-Plane equation works similarly to the Polar-Plot-Plane and the variable "x" is the angle of the arc on the unit circle measured in radians. For this Adventure, in order to make the angle/radians aspect very clear, the independent variable will henceforth be called θ . Also, $\sqrt{-1}$ will be used explicitly instead of the imaginary unit "***i***".

$$e^{\theta * \sqrt{-1}} \;=\; \cos\theta + (\sqrt{-1} \; * \sin\theta) \qquad \text{eq 13}$$

FIGURE 8



Complex Plane

It seems very strange, and hard to comprehend how a function that sweeps quickly up to infinity and slowly down to zero, as shown in Figure 2, can wrap itself into a perfect circle of radius one.

We are now ready for our quest: How does this happen? What does it mean?

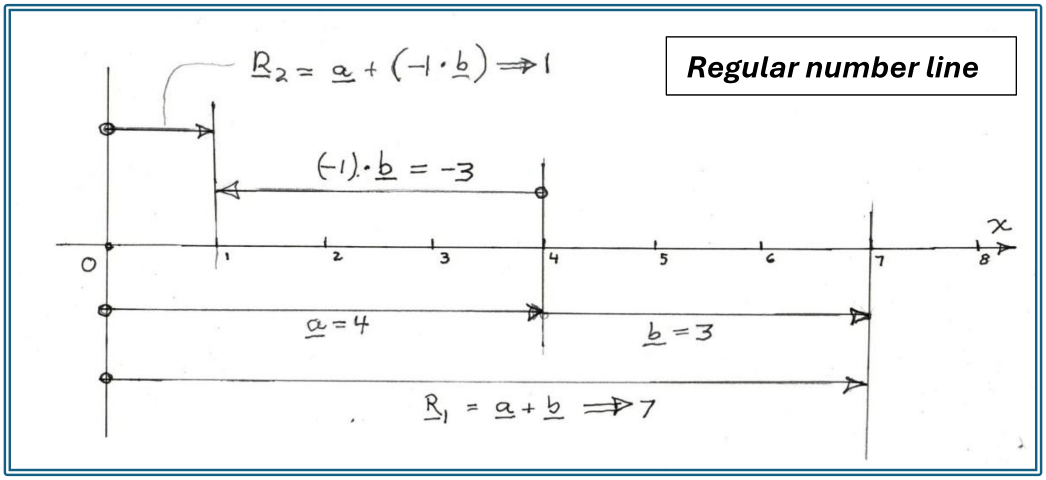
The Initial Detour:

Although Euler has already given us the answer, and the hint that it involves the fact that $i = \sqrt{-1}$, we have not yet asked the questions that give the answer meaning.

Let us turn our attention to $\sqrt{-1}$. There is no number on the regular number-line that serves this purpose. The regular number-line stretches from -∞ through zero to +∞ and includes all the numbers in between, including fractions and irrational numbers (those that can not be made up from simple fraction ratios) including "e" and "π." How, then, do we find $\sqrt{-1}$? If it is not on the regular number line, then perhaps it is something other, something off the regular number line.

Playing around, start with a simple vector based on evenly spaced numbers placed in order along the regular number-line as in Figure 9. One could draw an arrow from 0 to any point, say 4. That would give a vector "**a**" of length +4. Notionally, one could add another vector "**b**" of length +3, head to tail to the first vector and get a vector "$\mathbf{R}_1$" with a combined length of +7. Alternately one might <u>subtract</u> the vector "**b**" (length 3) from vector "**a**" (length 4). This is most easily visualized as reversing the direction vector "**b**" and again adding them head to tail, to get a new vector "$\mathbf{R}_2$" (length 1). See Figure 9. There is nothing new here, just establishing that we are all on the same page.

FIGURE 9



Note! We just have used simple addition together with an operator (multiplication by (-1)) to reverse the direction of vector "**b**". ***Let us be grandiose and say we "rotated vector 'b' by a half-turn on the paper."***

Furthermore, let us postulate a **new operator** that rotates a vector by one quarter of a turn (one quarter of a full circle) counterclockwise (ccw) on the plane of the paper. We will call the new operator "***q***" for quarter-turn, ***and that is all it does***.

Put this to work: modify (rotate) vector "**b**" by applying the operator "***q***" to get a new type of quantity, (***q*** ***b**) , which is no longer on regular number line, but now lies in the plane of the paper, which we will call the "***q***" Operator Plane. This new quantity points a quarter turn ccw from the original regular vector "**b**".

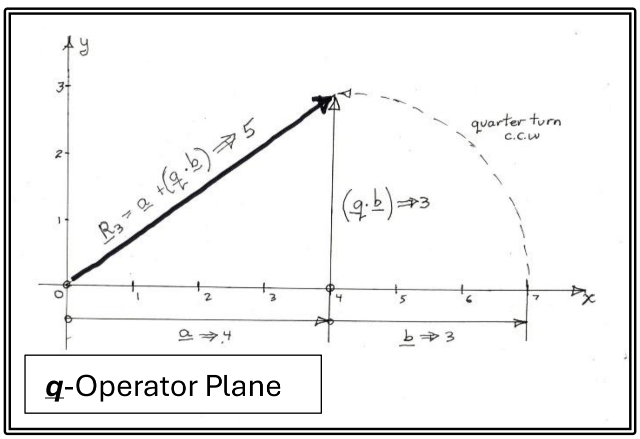
It would be desirable to combine this modified vector together with regular vectors in a meaningful way. The normal rules of plane geometry and trigonometry are useful in this regard.

Now add vector “**a**” together (head to tail) with the modified vector “**b**” (adjusted by the operator “***q*** **”)** to get new resultant vector $\mathbf{R}_3$

$$\mathbf{R}_3 = \mathbf{a} + (\boldsymbol{q} * \mathbf{b}) \qquad \text{eq 13}$$

and plot these vectors on paper in a “***q*** Operator Plane” as shown in Figure 10. We can plot vector “a” on the regular number line, and we can plot vector (***q*** ***b**)** as rotated ccw one quarter-turn. Then add them head to tail. Measure a resultant vector $\mathbf{R}_3$ of length of 5, with $\underline{R_3}$ pointed off the regular number line. We can specify the new vector $\mathbf{R}_3$ by its two components [**a**, ***q*** * **b**]. This is a new “other” type of quantity; it is not strictly limited to the regular numbers and number line. Check and see that adding them in any order will result in the same vector, $\mathbf{R}_3$.

FIGURE 10



Note, any vector quantity in this “***q*** Operator Plane” can be broken into components that project either onto the regular number line, or onto the perpendicular axis. In Figure 10 the perpendicular axis is shown as the “y” axis, or we might refer to it as the ***q***–rotated axis.

Now Euclid would have been happy with this construction, and Pythagoras would have approved since $\boldsymbol{a}^2 + \boldsymbol{b}^2 = (\boldsymbol{R}_3)^2$. The quarter-turn from operator **“*q*“** provides Pythagoras ‘ required right-angle. Not really much new here, let’s move right along with their blessings.

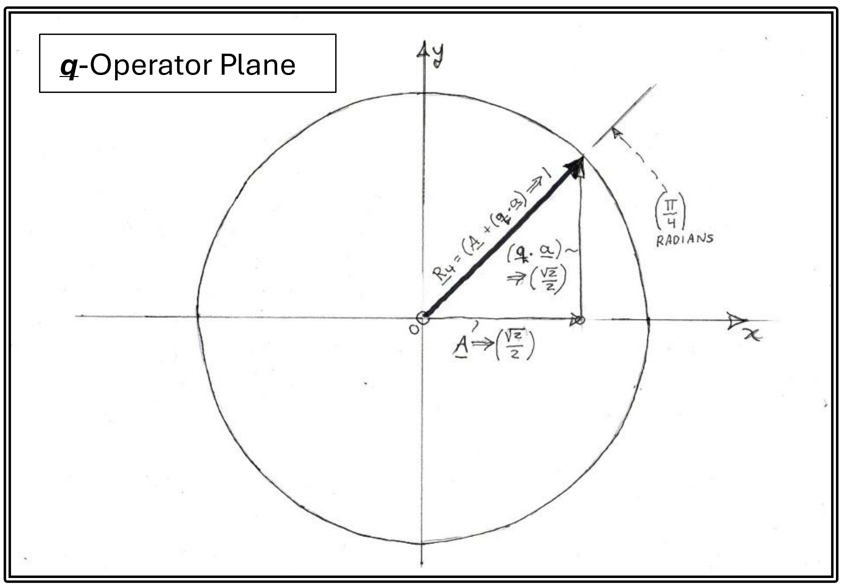
For this adventure we need to play a bit with this result and lay out a bit of algebra to confirm some general observations: For those so inclined most of the algebra is in appendices. Grab a pencil and paper to sketch these vectors and don’t let the algebra intimidate. In a short bit we will have some intuitive graphic approaches that will feel like guilty short cuts – but we will have the algebra in the appendices to justify them.

Play with these vectors:

Let's plot the sum of two new vectors: **A** = ((√2)/2) , and (***q*** * a), where **a** = ((√2)/2) as in Figure 11. The result will be a new vector **R4** with a magnitude of 1 pointing 45 degrees (π/4 radians) ccw from the real number line: We can specify it several equivalent ways:

- First, we know formula: **$R_4 = A + q * a$.**
- Secondly, we can specify its end coordinates: **R_4** ~» [(√2/2), ***q*** * (√2/2)].
- We can also use a polar form {radius (magnitude length), angle in radians}:
    - R_4 ~» **{1, π/4}**.

FIGURE 11



Now multiply **R_4** by operator ***q*** and see what happens … Should we expect an additional quarter-turn ccw? Let us check the general algebra:

$q * R_4 = q * (A + q * a) = q * A + (q * q) * a$ eq 14

Now if we interpret (***q*** * ***q***) as two quarter-turns, one after the other, then the resultant action is a half-circle turn (π radians) and the resultant is:

In rectangular coordinate form this plots to [-a, ***q*** * A],

In Polar coordinates this plots to {1, [(π/4) + (π/2)] } where the (π/2) is the quarter circle ccw rotation. The final resultant in polar coordinates = **{1, (3π/4)}.**

Let us check the numerical results with algebra detailed in the endnote[vii], substituting $(\sqrt{2})/2$ for the values of $\underline{A}$ and $\underline{a}$. Trace through the $\boldsymbol{q}$ rotations and interpret $(\boldsymbol{q} * \boldsymbol{q})$ as ***(-1)***

In rectangular coordinate form $(\boldsymbol{q} * \underline{R_4})$ plots to $(-(\sqrt{2})/2), \boldsymbol{q} * (\sqrt{2})/2)$.

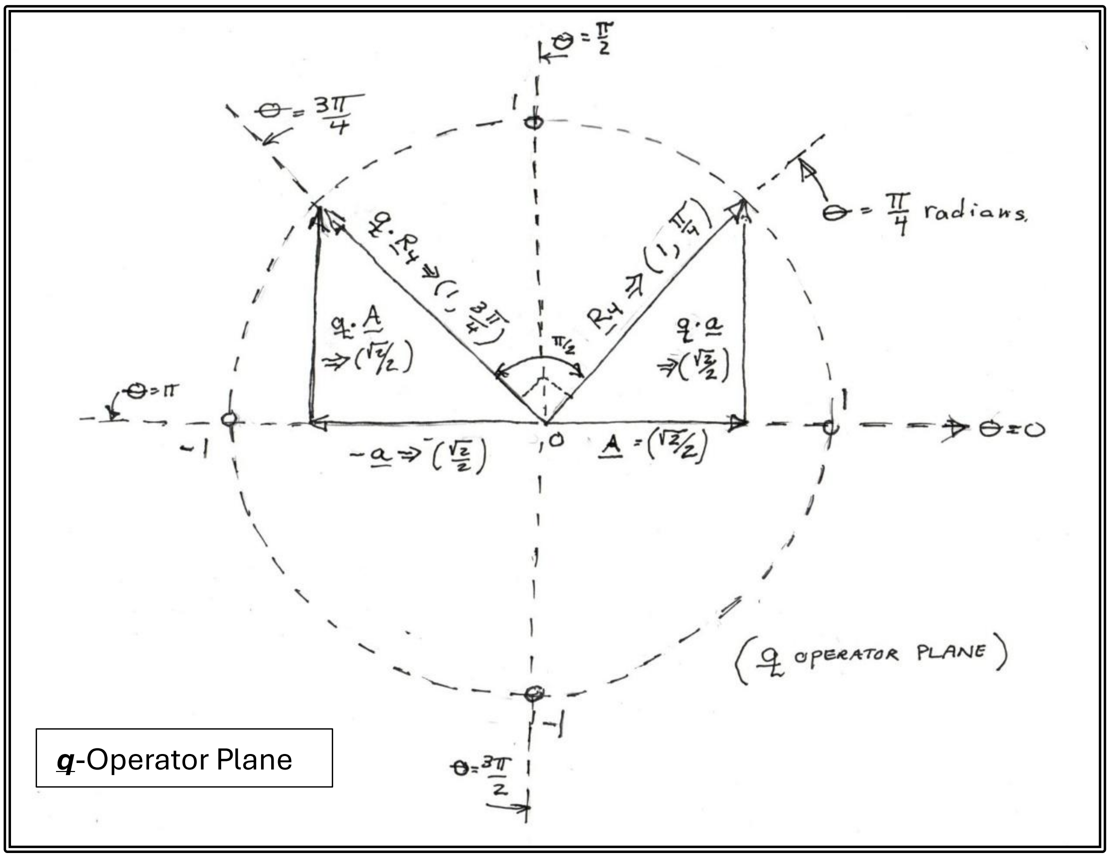
In Polar coordinates $(\boldsymbol{q} * \underline{R_4})$ plots to **{1, 3π/4}.**

This is shown in Fig 12, and confirms the expected quarter circle rotation of vector as long as we interpret $(\boldsymbol{q} * \boldsymbol{q})$ as two quarter-turns, so this implies that

$$(\boldsymbol{q} * \boldsymbol{q}) = (-1), \qquad \text{eq 15}$$

which makes operator $\boldsymbol{q}$, the quarter-turn ccw, operating suspiciously like the mysterious $\sqrt{(-1)}$.

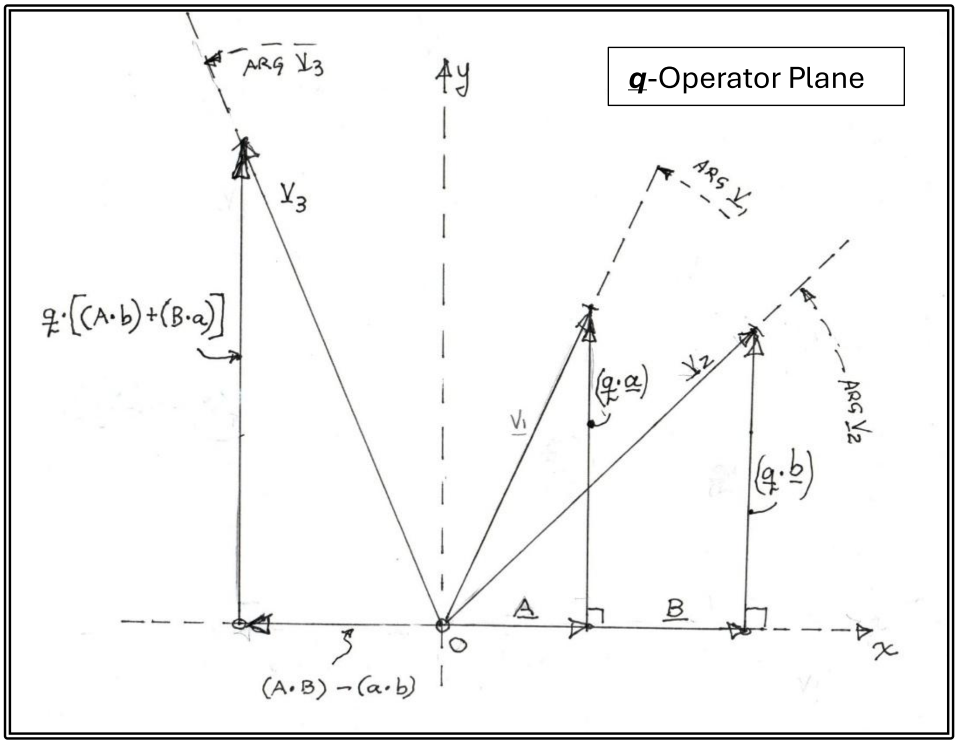
FIGURE 12



What happens when we multiply together two **general** vectors which include $\boldsymbol{q}$, -operator components? Let us find a new resultant:

$\underline{V_3} = \underline{V_1} * \underline{V_2}$ where $\underline{V_1} = (\underline{A} + \boldsymbol{q} * \underline{a})$, and $\underline{V_2} = (\underline{B} + \boldsymbol{q} * \underline{b})$ as shown in Figure 13.

FIGURE 13



Starting with the general algebraic expression for multiplying two arbitrary vectors:

$$\underline{V}_3 = (\underline{A} + \boldsymbol{q} * \underline{a}) * (\underline{B} + \boldsymbol{q} * \underline{b}) \qquad \text{eq 16}$$

Multiply this out as shown in end note [viii] and collecting "***q***" terms and letting ($\boldsymbol{q} * \boldsymbol{q}$) = (-1), we end up with:

$$\underline{V}_3 = [(\underline{A} * \underline{B}) - (\underline{a} * \underline{b})] + (\boldsymbol{q} * (\underline{A} * \underline{b} + \underline{B} * \underline{a})) \qquad \text{eq 17}$$

This is the general algebraic-solution of multiplying two arbitrary ***q***, -operator vectors. The algebraic-solution for resultant vector $\underline{V}_3$ is laid out graphically in Figure 13.

While that was fun, it would be very handy if the rumor of some graphical short-cuts for multiplying two vectors in the Complex Plane could work with our **q** Operator Plane vectors, namely:

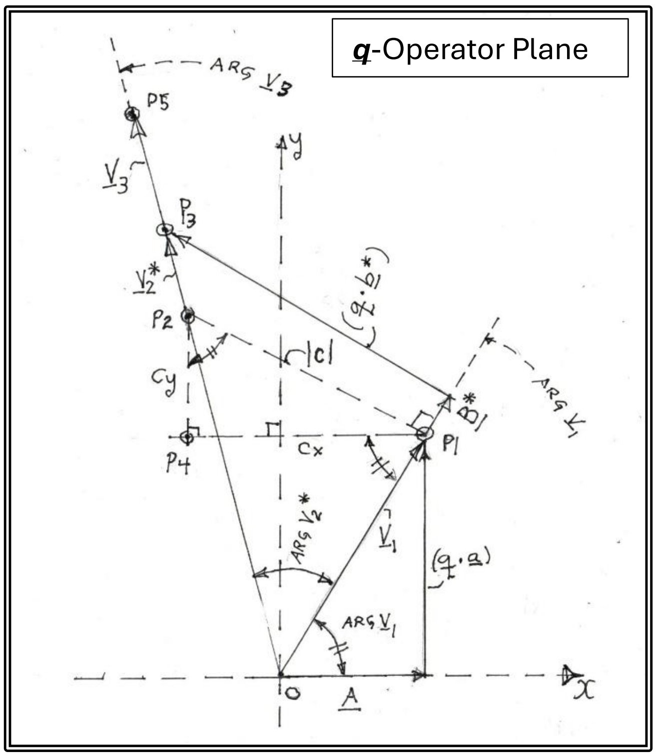
- Is it true that the magnitude of the resultant product-vector is equal to the product of the component vector magnitudes?
  - $|\underline{V}_3| = |\underline{V}_1| * |\underline{V}_2|$ eq 18
- Is it true that the argument (angle) of the resultant product-vector is the sum of the arguments of the component vectors?
  - $\text{Arg } \underline{V}_3 = \text{Arg } \underline{V}_1 + \text{Arg } \underline{V}_2$. eq 19

Let us check if this could be true with our **q** Operator Plane vectors. While algebra has been moved to Endnotes, It is worthwhile to be comfortable with these proofs.

Endnote IX [ix] explores the algebra to show that the magnitude of the resultant vector, $|V_3|$ matches the simple product of the magnitudes of the two initial vectors V_1, and V_2. This is a nice shortcut. Eq. 18 applies to **q** Operator Plane vectors.

Endnote X[x] involves some geometric arguments, shown in Figure 14, to prove the algebraic solution for the Argument of V3 matches the geometric angle from adding the arguments of V1 and V2. This is a very handy shortcut. Eq. 19 applies to **q** Operator Plane vectors.

FIGURE 14



END OF DETOUR

In the detour we postulated a special operator **q** which applies a quarter-circle ccw rotation to any vector.

- Vectors, including **q** components, can be added using the usual rules from geometry and trigonometry.
- When vectors in the q operator plane (including **q** components) are multiplied together they allow for two graphical shortcuts:

- The magnitude of the resultant product-vector is equal to the product of the component vector magnitudes:
  - $|\underline{V}_3| = |\underline{V}_1| * |\underline{V}_2|$
- The argument (angle) of the resultant product-vector is the sum of the arguments of the component vectors:
  - $\text{Arg}(\underline{V}_3) = \text{Arg}(\underline{V}_1) + \text{Arg}(\underline{V}_2)$

- Multiplied by itself (***q* q***) it provides a half-circle rotation, exactly like multiplication by (-1).
- Finally, ***q***, the quarter-turn ccw, is operating suspiciously like the mysterious √(-1).

The detour has provided a tool for use in our quest: How does the exponential wrap itself into the unit circle?

Playing with the Unit Circle:

In trigonometry the unit circle (radius = 1) is used to define the trigonometric functions, sine, cosine, tangent, etc. In mathematics the angles are measured in radians of arc length along the circumference. There are π radians in a semicircle, and 2*π radians in a full circle. Since the functions repeat with additional revolutions of the angle, the functions are tabulated for a single rotation of 0 to 2*π radians, or -π to +π radians, depending on the situation. However, the trigonometric functions do exist for rotations outside these ranges. The sine of 3*π just happens to be the same as the sine of π, etc, making these "non-unique" solutions.

We can observe that 1^1 ~» **{1, 0}** in polar notation and that $(-1)^1$ ~» **{1, π}** and √(-1) ~» **{1, π/2}** are all located on the unit circle. Furthermore, $(-1)^2$ happens to be 1, which is also on the unit circle at **{1, 2π}** in polar notation. Playing with this further:

A SPECIAL EXAMPLE CASE: Now Look at a special case using $\underline{R}_4$ = **{1, π/4}** from Figure 11. Find the new resultant $\underline{R}_6$ when $\underline{R}_4$ is multiplied by itself, that is $\underline{R}_6 = \underline{R}_4 * \underline{R}_4$.

Substitute in the numerical (rectangular) values for this case:

$\underline{R}_4 = [(\sqrt{2})/2\ ,\ \underline{q}^*((\sqrt{2})/2\ )]$

$\underline{R}_6 = \underline{R}_4 * \underline{R}_4$

Multiply it out:

$\underline{R}_6 = (\ ((\sqrt{2})/2\ )^2\ -\ ((\sqrt{2})/2\ )^2\ ) + \underline{q}_*\ (2_*\ (\sqrt{2})/2\ _*\ (\sqrt{2})/2\ )$, now simplify:

$\underline{R}_6 = 0 + \underline{q}*1$ which equals the unit vector rotated by a quarter turn ccw.

$\underline{R}_6 =$ **{1, π/2}** in polar form

It appears that we selected $\underline{R}_4$ to be ($\sqrt{\boldsymbol{q}}$) the square root of the $\boldsymbol{q}$ operator, which itself appears to be ($\sqrt{(-1)}$), so that would make $\underline{R}_4$ to be the fourth root of (-1), or $(-1)^{(1/4)}$.

A pattern starts to emerge. We could find the 16^{th} root of (-1) by dividing the angle/argument of (-1) into 16 pieces, that is π/16, so the $^{16}\sqrt{(-1)}$ ~» **{1, π/16}.** Even more generally:

$$^{n}\sqrt{(-1)} \sim\!\gg \{1, \pi/n\}. \qquad \text{eq 20}$$

Change now from the root notation to a more general exponent notation $^{n}\sqrt{(-1)} \sim\!\gg (-1)^{1/n}$.

We are now no longer limited to integers. For example,

$$(-1)^{3/2} = (-1)^{1/2} * (-1)^{1/2} * (-1)^{1/2} = ((-1)^{1/2})^3 = (-1)^{1.5}$$

Which maps to **{1, 3π/2}** in polar notation, a point on the unit circle. (Just add the angles!)

More generally, we can find any root or power of (-1) with a **mapping**[xi] (~») onto the unit circle:

$$(-1)^x \sim\!\gg \{1, \pi * x\} \qquad \text{eq 21}$$

We can reach any point on the unit circle by selecting an appropriate value for x, including multiple rotations around the unit circle.

In this context, values of x<1 are commonly thought of as “roots” and values of x>1 are commonly thought of as “powers.”

We also know that each of these points on the unit circle has a trigonometric representation. Sticking with the $\boldsymbol{q}$ operator notation:

$$\{1, \arg\} \sim\!\gg \cos(\arg) * \boldsymbol{q} \sin(\arg). \qquad \text{eq 22}$$

Or in the more usual complex number format:

$$\{1, \theta\} \sim\!\gg \cos(\theta) * \boldsymbol{i} \sin(\theta) = e^{i\theta} \qquad \text{eq 23}$$

And we arrive at an unexpectedly beautiful mapping:

$$(-1)^x \sim\!> \{1, \pi * x\} \sim\!\gg \cos(\pi * x) * \boldsymbol{i} \sin(\pi * x) = e^{i\pi * x} \qquad \text{eq 24}$$

$$(-1)^{x} \sim\!\!\gg e^{i\pi * x}$$

or,

$$(-1)^{x/\pi} \sim\!\!\gg e^{ix}$$

eq 25

This raises questions about meaning of the mapping (~») and the equivalence (=). If anything, our original question has now been compounded as we now have two beautiful exponential type functions mapping into the unit circle.

Oh wait! The function y(x) = 1 also maps to the unit circle as well! (y(x) = 1 could also be written in exponential form y(x)= b^0, where b is any number .... An infinite number of equations!)

We explore the polar mapping of the exponential equations further, but first some more notes about the "roots" of various numbers (vectors) in the **q** Operator Plane. Because of the cyclical nature of the angle/argument, there are multiple roots for any given vector. In fact, there are "n" distinct root vectors that could serve as the "nth" root of a particular vector, but they are not all the same, and there is a primary root which fits our classic understanding of the "root." As an easy example, the two square roots of the vector **{1, 0}** are 1= **{1, 0}** and (-1) = **{1, π }.** Now **{1, 0}** will square itself to **{1, 0}** while **{1, π}** will square itself to **{1, 2π}.** While we would consider **{1, 0}** to be the primary root, it should be noted that either (+1) or (- 1) would be considered the two "solutions" for the quadratic equation:

$$x^2 - 1 = 0.$$

Thinking in polar notation and visualizing the polar short-cut rules can help with understanding these "roots." Take the 5^{th} root of (-1) as an example. There will be five distinct vectors in the ***q*** Operator Plane. which will land on (-1) when they are raised to the fifth power. They are:

- **{1, π/5}.** The primary root will land on **{1, π}** when raised to the fifth power.
- **{1, 3π/5}.** Which will land on **{1, 3π}** when raised to the fifth power. But **3π** appears the same as **π** because the second half lap around the circle plots directly over the first lap.
- Similarly **{1, 5π/5}** will land on **{1, 5π}** with two and a half laps around the circle...
- Similarly **{1, 7π/5}** will land on **{1, 7π}** with three and a half laps around the circle...
- And Finally **{1, 9π/5}** will land on **{1, 9π}** with four and a half laps around the circle...

- Any additional root-like vectors, say **{1, 11π/5}** will continue the pattern, but they themselves will not be distinct from the previous n root vectors. For example **{1, 11π/5}** will plot the same as **{1, π/5}.**

Although the example used (-1), a number off the regular number line, this pattern also works for arbitrary vectors in the ***q*** Operator Plane. There will be n distinct root vectors in the ***q*** Operator Plane that could serve as the "nth" root of a particular vector. A bit of exploration reveals that the n roots are:

For m=1,2…n (m = 1 is the primary root)

$$\text{Root}_m = \{\, |\underline{V}|^{1/n},\ \ ((\text{Arg}\underline{V})/n)(1+2(m-1))\}. \qquad \text{eq 26}$$

Where |V| indicates the magnitude of vector V.

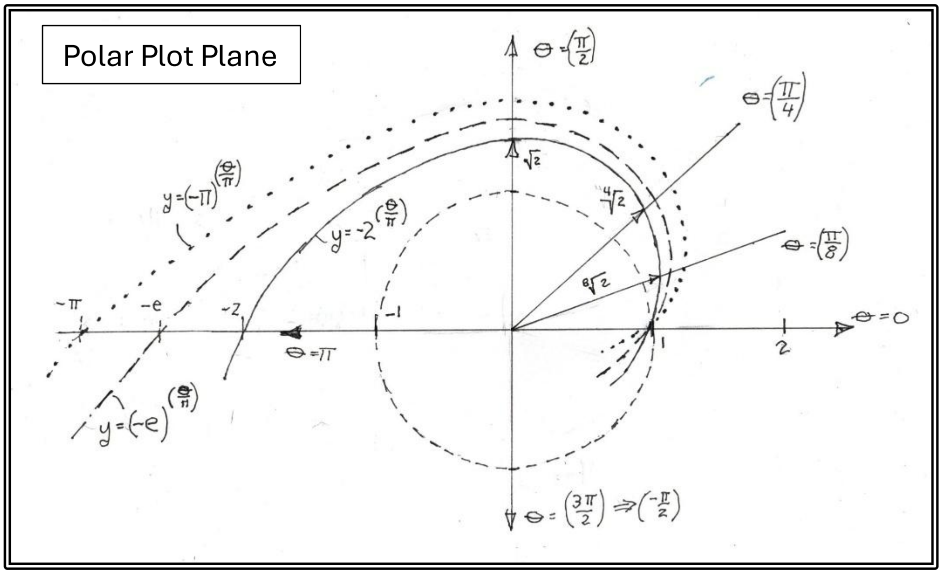
We are usually concerned mainly with the "primary" root, but it is worth remembering that the primary root is one of "n" distinct roots that could serve as the "n^{th} root" of any vector in the "***q*** operator plane," and this also includes all numbers on the regular number line.

BEYOND THE UNIT CIRCLE:

So far we have considered only the roots of (-1) and all those solutions lie on the unit circle so that when the roots are multiplied together they stay on the unit circle because the magnitudes are all (1) and $(1)^x$ is still magnitude 1, by both algebra and by the Polar Shortcut.

What happens when we explore the other points off the unit circle. Let's consider, for example $(-2)^{(\theta/\pi)}$. The polar vector to the point (-2) on the regular number line is **{2, π }.** Let's consider the vector to √(-2); using the "Polar Shortcut" rules: what Argument would get to π when added to itself? Clearly (π/2). And what magnitude would get to 2 when multiplied by itself? Clearly √(2). Therefore, we have a new vector **{√2, π/2)** which acts as square root of (-2). This vector is plotted in Figure 15 along with the 4^{th} root of (-2) which is **{$2^{1/4}$, π/4}.**

FIGURE 15



In fact, we will find that all the square roots of negative numbers have an argument of **π/2.**Similarly, we will find that all the primary fourth roots of negative numbers are along the **π/4** argument and have magnitudes equal to the fourth root of the absolute value of the negative number. Continuing this process, we will find that the roots of (-2) follow a spiral pattern as shown in Fig 15,

The very similar spirals for the roots of $(-e)^{(\theta/\pi)}$ and $(-\pi)^{(\theta/\pi)}$ are also shown in Figure 15. Note that each of the spirals approach the point **{1, 0}** as x approaches 0. This makes sense since $X^0 = 1$, for all X.

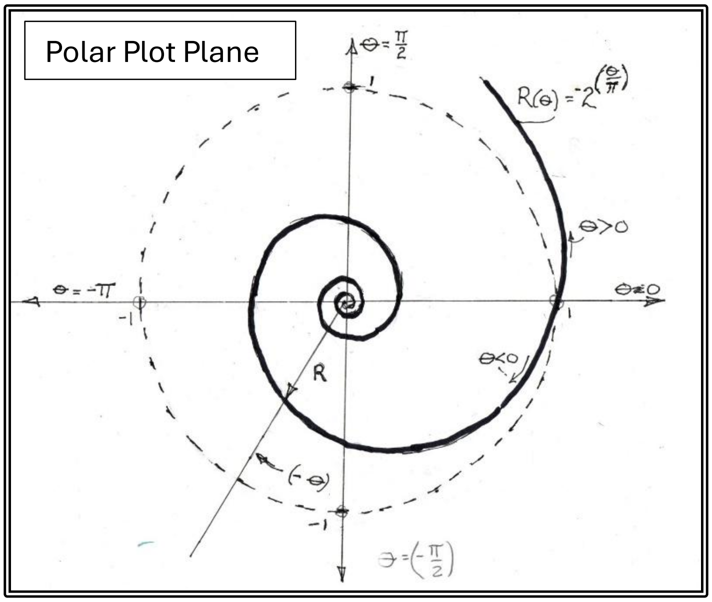
By now there should be some comfort, and even, perhaps, intuition with applying the polar vector multiplication rules to find roots and powers of any vectors, including any negative numbers and composite numbers made of real-number vectors added to rotated vectors.

NEGATIVE EXPONENTS

Numbers with negative exponents can also be expressed in the Polar Plane. While θ>0 indicates a positive (CCW) angle, any θ<0 indicates a CLOCKWISE (CW) angle, completely paralleling the reversal in direction for vectors multiplied by (-1). Then $b^{(-\theta)} = 1/(b^{(|\theta|)})$ using the normal rules of algebraic notation. The plot of $b^{(-\theta)}$ will be INSIDE the unit circle, and will spiral in a clockwise direction approaching the 0-radius point in an infinite number of turns. For example, $\underline{\mathbf{P}}(\theta) = e^{-\theta} = \{\, 1/(|e|^{|\theta|}),\ -\theta \,\}$.

Figure 16 shows the inwardly spiraling function $(-2)^{(-\frac{-\theta}{\pi})}$, note that the scaling by (1/π) slows the rate of the inward spiral. **A**s an example, the vector R shown in Fig 16 has an angle of approximately (-3π/4) and has a length of about 0.59 = $\left\{ 1/(|-2|^{\left|\frac{\frac{-3\pi}{4}}{\pi}\right|}) \right\}$ = $\{ 1/(2^{(3/4)}) \} = (2^{(-3/4)})$.

FIGURE 16



Vectors with negative arguments or angles can be directly added to those without negative angles. Multiplication of vectors with negative angles is also straight forward, the negative angles move things CW instead of CCW, and the vector magnitudes are absolute values in both positive and negative angle vectors.

NEXT STEPS ON THE ADVENTURE:

The ***q*** operator has been quite successful in manipulating vectors in the polar plane to find roots to negative numbers, etc., imitating √-1.

Following Euler's example, try using it with the Taylor Expansion for $e^{x\sqrt{-1}}$ substituting the ***q*** operator where Euler used i =√-1.

$$e^{qx} = 1 + \frac{qx}{1!} + \frac{(qx)^2}{2!} + \frac{(qx)^3}{3!} + \frac{(qx)^4}{4!} + \frac{(qx)^5}{5!} + \frac{(qx)^6}{6!} + \frac{(qx)^7}{7!} + \frac{(qx)^8}{8!} + \frac{(qx)^9}{9!} \ldots, \; -\pi < x < +\pi$$

Modified eq 9

$$e^{\boldsymbol{q}\,x} = 1 + \boldsymbol{q}\,\frac{x}{1!} - \frac{(x)^2}{2!} - \boldsymbol{q}\,\frac{(x)^3}{3!} + \frac{(x)^4}{4!} + \boldsymbol{q}\,\frac{(x)^5}{5!} - \frac{(x)^6}{6!} - \boldsymbol{q}\,\frac{(x)^7}{7!} + \frac{(x)^8}{8!} + \boldsymbol{q}\,\frac{(x)^9}{9!} \ldots,$$

Modified Eq 10

Note (1! = 1). However (9! = 9*8*7*6*5*4*3*2*1= 362,880) so the higher terms quickly get very small and almost meaningless for practical purposes.

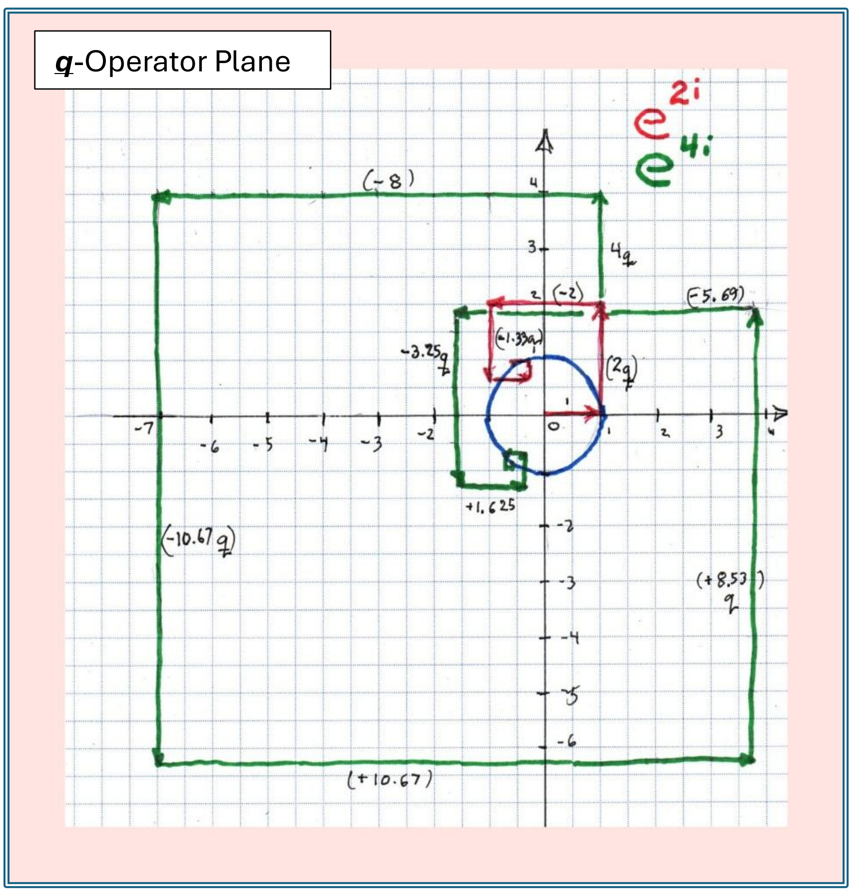
Figure 17 shows a plot of this series sum of vectors from Modified Equation 10 term by term using specific values of x.

For x=2 the vectors are plotted in red and the sums to 9 terms are:

$e^{2\boldsymbol{q}} = 1 + 2\underline{\boldsymbol{q}} - 2 - 1.33\underline{\boldsymbol{q}} + 0.667 + 0.2667\underline{\boldsymbol{q}} - 0.889 - 0.0254\underline{\boldsymbol{q}} + 0.0063 + 0.0014\underline{\boldsymbol{q}} - \ldots$

Adding these up to 17 terms and the net sum is [-0.4166 + 0.9093$\underline{\boldsymbol{q}}$] = {1, 2 rads} right on the unit circle (blue). Interestingly, if you add the absolute value of each vector length the sum after 17 terms is 7.3891, which is identical with e^2.

FIGURE 17



Plotting in green for x=4, the vectors sums to 9 terms are:

1 + 4***q*** – 8 - 10.6667***q*** +10.6667 + 8.5333***q*** - 5.6889 -3.2508***q*** +1.6254 +0.7224***q*** - …

Adding these up to 17 terms and the net sum is [-0.6536 - 0.7568***q***] = {1, 4rads}. Interestingly, if you add the absolute value of each vector length the sum after 17 terms is 54,5981, which is within one unit of e^4. Even this tiny discrepancy would disappear with the inclusion of additional terms.

Figure 17 shows how the ***q*** operator rotates the terms of the Taylor expansion, term by term to create the unit circle mapping of e^{qx}. Each additional term represents an additional derivative. One can suspect that the mystery of e^{ix} follows a similar pattern. Counting from 0, the odd terms contribute to the q rotated vectors that create the "sine" component of the total. The even terms contribute to the horizontal elements that create the cosine component of the total. And the total length of $|e^x|$ is distributed among the component vectors. Very interesting!

Is the quest complete? Well, *how is our intuition?*

My own intuition felt pretty comfortable with most of this, but then predicted that 2^{qx}, and π^{qx} would probably plot to circles; smaller than the unit circle for 2^{qx} and larger than the unit circle for π^{qx}. After all, e^x is pretty special and might have unique claim to the unit circle. **NOT SO!**

2^{qx} plots to the unit circle, but angle is reduced. π^{qx} plots to the unit circle but the angle is increased. That effect was actually predicted by the scaling equation in the endnote (ii) on scaling of logarithmic spirals:

$$b^{\theta} = e^{\theta * \ln(b)}.$$

That is, all the exponential-type equations are (angle)-scaled versions of e^x !

While intuition is pretty valuable, algebra seems more immutable. I regret that I seemed to have missed this intuition lecture, way back when. I think it would have been helpful in a career that frequently used exponentials and the complex plane for vibrations, ship motions, electrical equipment design, electronics and control systems.

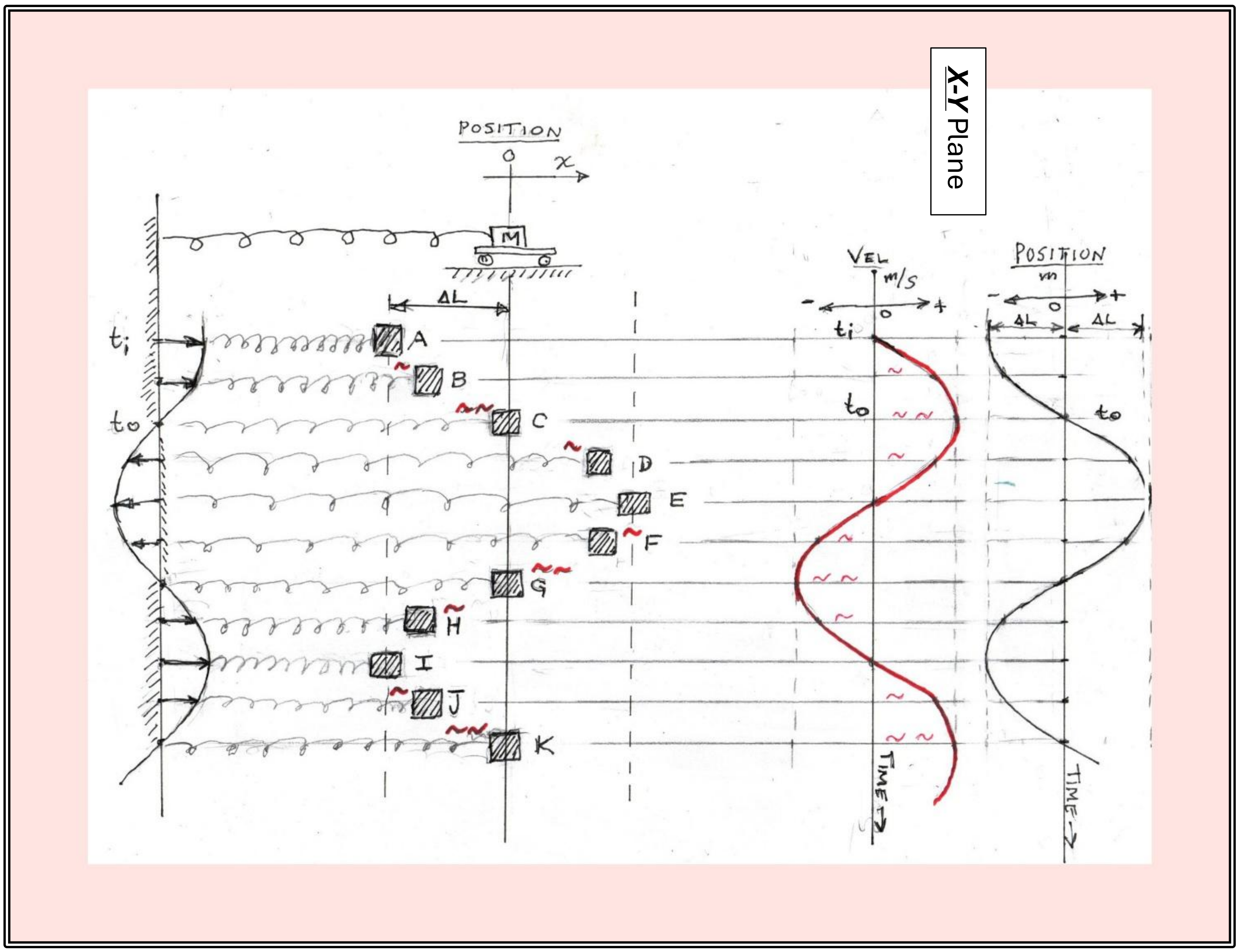
Hats off to Euler and his peers!

ONWARD!

Let the adventure continue to exercise intuition with an easy practical example. Show how exponentials and unit circles combine to predict the vibration response of a mass M attached to a spring.

In Figure 18 a mass is attached to a horizontal spring in its neutral position, so the spring puts no force on the mass. The other end of the spring is fixed at a wall. The mass is on a frictionless cart so ignore any gravity and just consider the horizontal spring forces. Let some external agent move mass a length ΔL to the left, held there for a moment. Then let the mass be released at initial time t_i. How will the mass respond? Watch closely the energy in the system.

Figure 18 - Mass and Spring System



At this first moment, just after the spring is moved, the spring has a compressive force pushing the mass to the right. The mass is not yet moving but intuition should tell us that, once the mass is released, the mass will start to accelerate to the right in response to the spring force. The initial spring force will be F = k*ΔL, where k is the spring constant in force (N) per unit length (m), or [N/m]. At this point the spring has a stored "potential" energy of $PE_i = (1/2) * k * (\Delta L)^2$. Also, at this point the mass has no kinetic energy due to motion, $KE_i = 0$. Therefore, the total energy at initial time, t_i , is :

$$ET_i = PE_i + KE_i \, .$$

A fixed dose of energy was added to the system when the spring was suddenly compressed. But now, with a frictionless cart, the total system energy will remain constant. As the mass starts to move to the right it will gain kinetic energy $KE(t) = (1/2)* M * [v(t)]^2$, but the spring compression will be released as the mass moves to the right, so the stored potential energy will be $PE(t) = k* x(t)^2$. The total energy will stay fixed ($ET(t)=ET_i$), so the kinetic energy gained by the mass will match the reduction in stored spring energy. If the easily measured mass, M, and the spring constant, k, are known, then we can calculate the velocity v(t) for any position x(t).

Select some convenient values for spring constant k, mass M and initial stretch length ΔL.

- $K = 2$ N/m
- $M = 2$ kg
- $\Delta L = 1$ m

In the initial position (A) the compressed spring will exert a force to the right on the stationary mass (time t_i). A fixed amount of potential energy has been added to the system by whatever agent did the compression work. That potential energy can be calculated

$$PE_i = ½ *k * (\Delta L)^2 = 1 \text{ N*m} = 1 \text{ joule}$$

The mass is motionless at the initial time t_i so the kinetic energy $KE_i = 0$. The total energy of the system at any point is the sum of the potential energy and the kinetic energy. Since we have no losses (friction) the total energy will stay constant, so:

$$TE(t) = PE(t) + KE(t) = 1 \text{ joule}$$

Beginning at position A, the mass will start to accelerate to the right in response to the spring force. As the mass moves from its initial position the spring compression is going down, the mass will accelerate less quickly as it moves further to the right. In fact, when the mass has moved a distance $x(t) = \Delta L$ (position C) the spring will be back in its neutral position and the spring force will be zero. The stored energy in the spring will be zero at this point. However, at this point the mass is moving to the right, now with momentum, and the kinetic energy will equal the (constant) total energy of the system, ET_i.

For purposes of this illustration, call this moment, at which the mass has returned to the initial starting point (C), time zero (t_0). Spring forces are shown in Figure 18 as arrows at the far left, where the spring attaches to the wall.

Because there is no spring energy at this instant, t_0, the velocity can be calculated from the kinetic energy:

$$KE_0 = ET_i = 1 \text{ joule } = ½*M*(v_0)^2; \quad \text{So } v_0 = 1 \text{ m/s}$$

Starting at t_0 the motion to the right will start to stretch the spring and provide a force pulling to the left, acting to slow the velocity to the right. The mass will start to slow down as it moves to further the right. This deceleration will start slowly (because the spring forces are small when the extension first starts) but will become stronger and stronger until the mass comes to a stop at point E. The energy will have shifted back from the kinetic energy into (tension) potential energy stored in the spring again. Solving for the total potential energy, it turns out that the mass comes to rest after having moved 2* ΔL, or to a position x=+ΔL to the right of the neutral position (x=0).

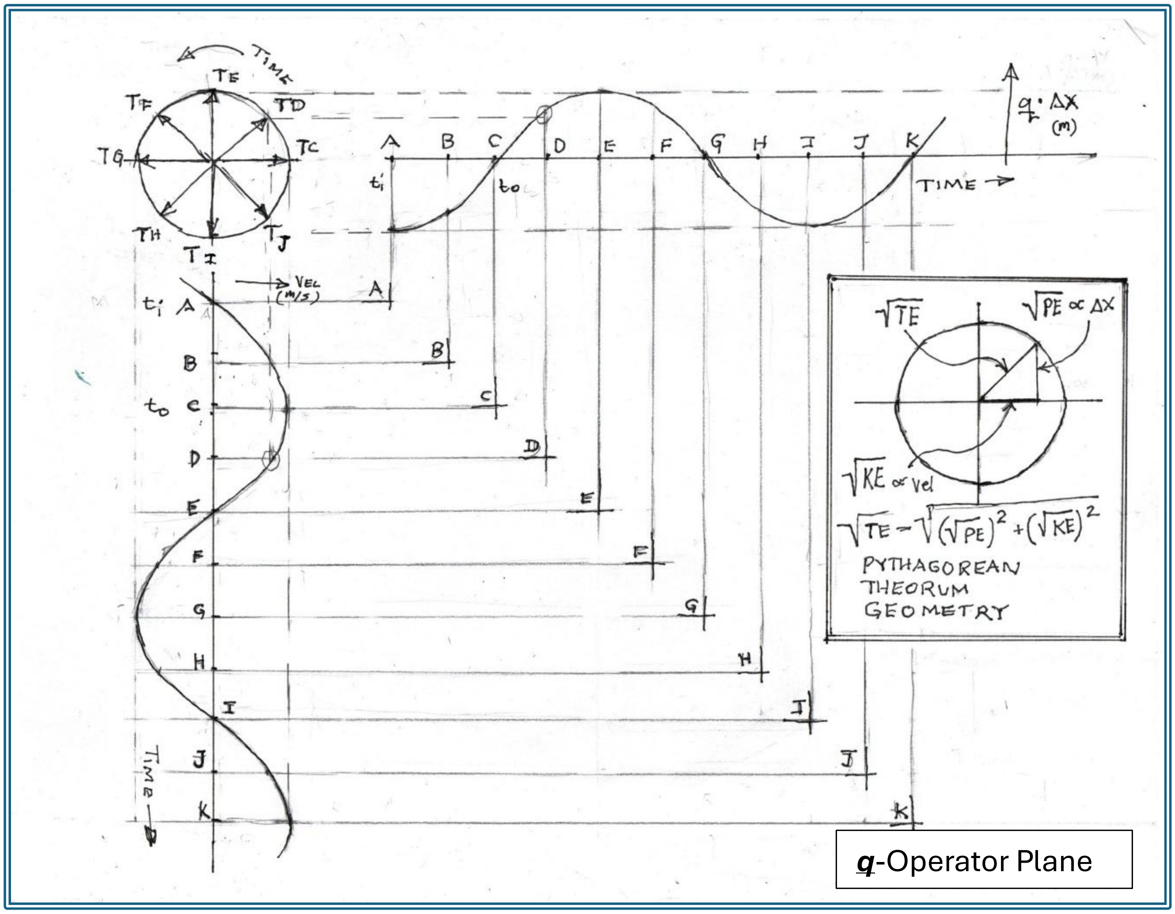
At this point the process reverses, now accelerating the mass to the left. The mass shuttles back and forth between the initial position on the left [x (t) = -ΔL], and the extreme point on the right, [x(t) = (+ΔL).] The energy shuttles back and forth between potential energy in the spring at the extreme points of the travel and kinetic energy in the mass, which peaks at the center of the vibration, wherever [x(t) = 0.]

Intuition told you all this and probably suggested that the shuttling motion might be sinusoidal. Perhaps intuition can prove that this motion is sinusoidal?

Figure 18 also shows the Δx position plotted against time (running down vertically) in the far right of the figure. A related plot of velocity (red curve) is just to the left of the Δx position plot. While these vertical "time strip" plots are a bit unusual, they mimic the normal X-Y-plane sine and cosine curves shown back in figure 7. Time t has been substituted for independent variable x.

Now for an important shift in perspective: Figure 19 remaps the velocity and x curves and shows them feeding into the familiar unit circle. See that the velocity of each point (A, B, C...) maps up into the cosine of the corresponding angles (T_A, T_B, T_C...). Similarly, the Δx of each point (now plotted rotated a quarter turn CCW) maps into the $\boldsymbol{q}$-rotated sine of the corresponding angles in the unit circle. Time t is being plotted as angle, starting t=0 (for (TC)) on the horizontal axis.

FIGURE 19



And now for energy: The kinetic energy, KE(t), is proportional to the velocity squared. KE(t) = $\frac{1}{2}$ M $v(t)^2$. The potential energy in the spring, PE(t), is proportional to the Δx squared. PE(t) =$\frac{1}{2}$ k $x(t)^2$. The total energy is sum of the two:

$$ET(t) = \tfrac{1}{2} M\, v(t)^2 \quad + \quad \tfrac{1}{2} k\, x(t)^2$$

That is exactly what the unit circle in Figure19 is showing: the composite vector on the unit circle, at each point is the (square root) of the total energy, which is a constant. The sine (Δx) component is the square root of the stored potential spring energy (plotted with the quarter turn rotation, ***q***). Meanwhile the cosine (velocity) component is the square root of the kinetic energy stored in momentum.

The requirement that the total energy stay constant dictates that motion, displacement (Δx) and the velocity work into the unit circle, or sinusoidal motion, where velocity follows the cosine curve and (Δx) follows the sine curve. The total energy stays constant while the form of the energy cycles back and forth between potential spring energy and kinetic motion energy. Intuition has backed-us into this corner in which the motion follows this path around the unit circle[xii] that totally meshes with Euler's formula:

$$e^{\theta*\sqrt{-1}} = \cos\theta + (\sqrt{-1} * \sin\theta) \qquad \text{eq 13 (repeated)}$$

This intuition adventure has used the quarter turn operator, ***q*** , to stress the geometric interpretation instead of the more algebraic $\sqrt{-1}$, or even more conventionally, ***"i"*** (or "*j*" for electrical engineers), but the results are the same.

THE NEXT STEP:

Now watch what happens over time if we add a slow bleed of energy (slight friction) to the spring mass system. Intuitively, we expect the system response to die down over time.

Figure 20 adds an additional perspective shift with time adding a new dimension to our graph. We have the ***q***-Operator Plane, as before, now augmented into a cylindrical coordinate system with time running down the length of the cylinder. The sketch shows a diminishing spiral which reflects the anticipated gradual subsidence of the system response as time progresses along the diagonal. The sinusoidal response continues at the same rhythm (period "T" is constant) but with less amplitude as time goes on. In fact, the sinusoidal response is now bounded by a diminishing exponential (shown in red).

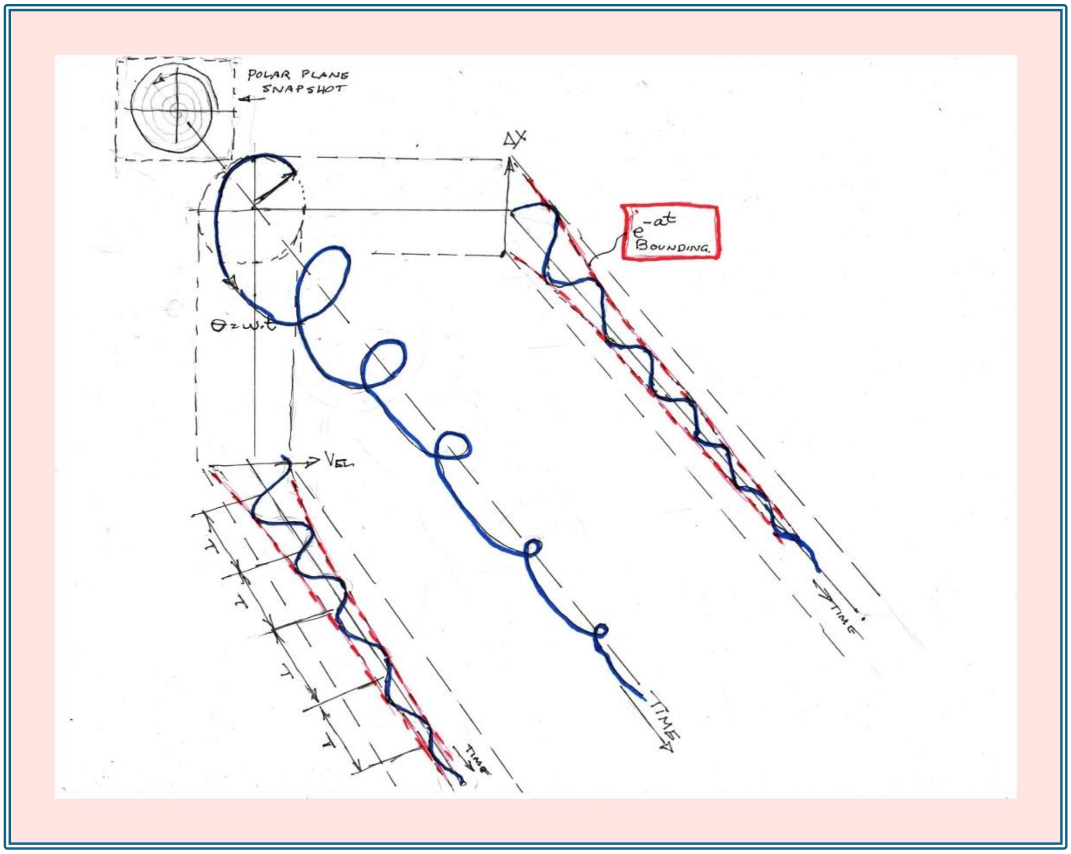
Mathematically we can extend the polar equation into this 3-dimensional plot since the angle, Θ, is growing linearly with time: Θ = ω*t.

$$\underline{\boldsymbol{R}}\{\Theta,t\} = \underline{\boldsymbol{R}}\{(\omega*t), t\} = (e^{q\omega t})*(e^{-at}) \qquad \text{eq 27}$$

Where ($e^{q\omega t}$) is the underlying sinusoidal oscillation of Figure 17and (e^{-at}) is the diminishing exponential, shown in red, that bounds the response as shown in Figure 20.

The velocity components and displacement x components are shown in XY time projections. These show the oscillation of the energy back and forth from kinetic energy $(\text{velocity})^2$ to potential energy $(\Delta x)^2$.

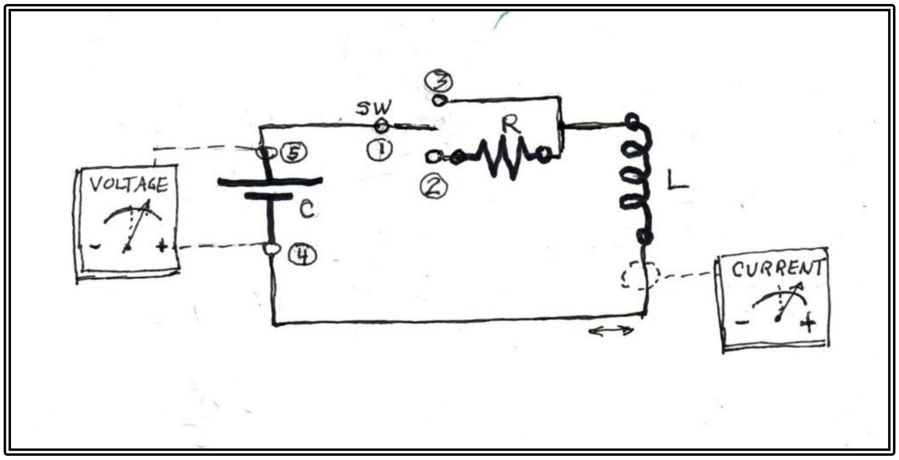
FIGURE 20



Finally, a similar simple electrical system, the resonant LCR tank circuit, is shown in Figure 21. In this circuit electrical energy will shift back and forth between inductive (kinetic) energy in the magnetic field and potential electrostatic energy stored in the capacitor voltage.

When the switch is closed between terminals 1 and 3, any initial voltage stored in capacitor C will push on the inductor L and start to make current flow. With only slight re-labeling the response curve of figures 18 and 19 will describe the response of this electric circuit. Just substitute capacitor volage V(t) for displacement Δx(t), and substitute inductor current I(t) where we had velocity Vel(t). Furthermore, if a small loss is added to the circuit (switch terminal 1 to connect with terminal 2 and include small resistance R in the circuit) the energy will slowly dissipate, and the response will die out, just as shown in Figure 20 for the spring system.

FIGURE 21



A FINAL NOTE:

These polar plot perspectives are useful for thinking about the response of these systems because the independent variable, "x", or "Θ," or time "t" is in the exponent; therefore algebraically we use the "**law of exponents"** to combine equations. The law of exponents (using base B, for example):

$$B^a * B^b = B^{(a+b)}$$

When we multiply the quantities made from bases, we add the exponents, just like we add the angles when we multiply polar vectors. ***So, the polar plot graphics rules for vectors mimic the algebraic rules for exponential type quantities***, B^x, or e^x or even π^x. The ***q*** operator allows us to create combined vectors (beyond the regular numbers) that follow the polar graphic rules for vectors and mimic the algebraic rules for exponential type quantities including the sinusoidal trigonometric functions.

Important Note! One could plot other functions, like x^2 in a polar plot; $R(x)= x^2$, and get a spiraling curve in the polar plot plane, but it's meaning would be unclear because the independent variable "x" is not in the exponent, so exponent algebra would not apply.

SO:

Yes, The ***q*** operator seems suspiciously like the imaginary unit "***i***".

Doesn't that moon snail volute sketched on the title page call out for exponential spirals?[xiii]

This adventure may have been a fool's errand [xiv], but I think I am better for it…

ENDNOTES

[i] **Endnote I: "The most beautiful formula in Mathematics":**

Leonhard Euler (1707-1783):

$$e^{i\pi} + 1 = 0$$

"…perhaps the most compact and famous of all formulas," - Edward Kasner and James Newman, *Mathematics and the Imagination* (1940) as quoted by Eli Maor, *e The Story of a Number* (1994)

"…surely ranks among the most beautiful formulas in all of mathematics,"- Eli Maor, *e The Story of a Number* (1994)

Roger Coats (1682-1716) and Abraham DeMoivre (1667-1754) each discovered equivalent relations prior to Euler's arriving at this formulation. - Eli Maor, *e The Story of a Number* (1994)

Note: Other formulas have been nominated for this honorific. For example, see:
https://www.youtube.com/watch?v=fsLh-NYhOoU&t=34s

[ii] **Endnote II: Scaling of Logarithmic Spirals**
The exponential type equations $R(\theta) = b^{\theta}$ are all very similar, all pass through the point {1,0} and vary, primarily, in the intersection angle between the function and the ray coming out from the 0 point. (Note "b" refers to the exponential base for example $2^{\theta}, \pi^{\theta}, etc.$) Interestingly, each spiral has a characteristic intersection angle that is the same for any ray coming out from the point 0, all the way around the circle, earning the logarithmic spiral the nickname, the "equiangular spiral". For the classic $R(\theta) = e^{\theta}$ equation that intersection angle is π/4 which corresponds to a slope of 1 at (x=0) in the XY plot. In many books the graph of the classic equation incorrectly shows an intersection angle nearly perpendicular to the ray from 0. The exponential type equations can be scaled from one base to another by the following formula:

$$b^{\theta} = e^{\theta * \ln(b)}$$

NOTE TO SELF:

How do you explain the spiral polar plot graph of $(-b)^{\theta}$ since there are no natural logarithms for negative numbers? The polar plot instructions get around this by specifying to plot the magnitude of the expression at the polar angel Θ:

$$(-b)^{\theta} = \left|e^{\theta * \ln(|b|)}\right| * e^{\theta * \sqrt{(-1)}}$$

This is essentially the Euler equation vector angle from the unit circle, multiplied by the real number magnitude of the expression. Note that the scaling factor ***(ln(|b|)*** does not appear in the Euler angle, but only in the magnitude portion of the vector.

[iii] **Endnote III: Golden Rectangles for the Golden Logarithmic Spiral**
The golden rectangle has specific proportions of the golden ratio of length to width, φ ~ (1:1.618…). The rectangles are fascinating because the rectangle can endlessly grow or be divided into components that have the same proportions. It grows by appending a square the dimension of the length. It shrinks by subtracting a square the dimension of the width. See the Wikipedia link: https://en.wikipedia.org/wiki/Golden_ratio

[iv] **Endnote IV**: **Fibonacci Sequence**
The Fibonacci Sequence is an infinite series of integers 0, 1, 1, 2, 3, 5, 8, 13, 21, 34 ... where each new entry is the sum of the previous two entries. It is a sort of integer "quantized" model of the exponential function where instantaneous growth is proportional to the current value. It is famous for its approximation of many natural processes. See the Wikipedia link:
https://en.wikipedia.org/wiki/Fibonacci_sequence

[v] **Endnote V**: **Taylor Series approximations**
Taylor Series approximations fit an infinite series of power functions in the XY-plane to a given function using the local derivatives of the function. It is instructive to watch the animation of the approximation process. Use the link:
https://simple.wikipedia.org/wiki/Taylor_series

[vi] **Endnote VI**: **Taylor Series for Trigonometric Functions:** (from Wikipedia)

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

These Tayor Series equations are derived from fitting the XY-plane sine and cosine functions. It is instructive to see the approximation process to the cosine function. Use the link:
https://www.wolframalpha.com/input/?i=taylor+series+cos+x

[vii] **Endnote VII**: **Algebra of *q*-operator rotation for example vector: R4 = {1, π/4}**

$\boldsymbol{q} * R4 = \boldsymbol{q} * [(\sqrt{2})/2 + \boldsymbol{q} * ((\sqrt{2})/2)]$

$\boldsymbol{q} * R4 = [\boldsymbol{q} * (\sqrt{2})/2 + (\boldsymbol{q} * \boldsymbol{q} * ((\sqrt{2})/2))]$.

$\boldsymbol{q} * R4 = [\boldsymbol{q} * (\sqrt{2})/2 + (-1 * ((\sqrt{2})/2))]$, assuming $\boldsymbol{q} * \boldsymbol{q} = -1$, or *(two quarter turns = ½ turn)*

[viii] **Endnote VIII**: **Algebra of Multiplication of two general vectors in the *q*-operator plane.**

Starting with the general algebraic expression for multiplying two arbitrary vectors:

$V_3 = (A + (\boldsymbol{q} * a)) * (B + (\boldsymbol{q} * b))$,

Multiply this out and then collecting "***q***" terms and letting $(\boldsymbol{q} * \boldsymbol{q}) = (-1)$, we end up with:

$$V_3 = (A * B) + A * (\boldsymbol{q} * b) + (\boldsymbol{q} * a) * B + (\boldsymbol{q} * a) * (\boldsymbol{q} * b),$$

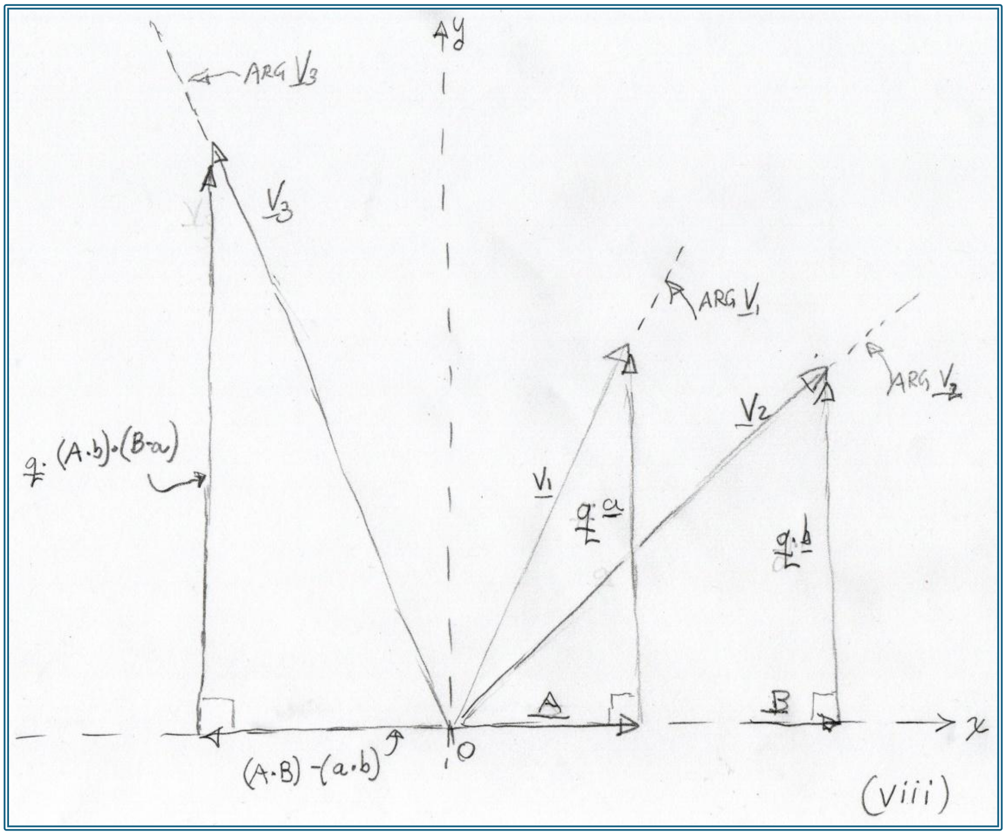
$$V_3 = (A * B) + (\boldsymbol{q} * \boldsymbol{q} * (a * b)) + (\boldsymbol{q} * (A * b + B * a))$$

$$V_3 = [(A * B) - (a * b)] + (\boldsymbol{q} * (A * b + B * a))$$

This is the general algebraic-solution of multiplying two arbitrary ***q***-operator vectors.

[ix] **Endnote IX: Checking out the Magnitude Short Cut:**

FIGURE 13 (repeated) ($\underline{q}$-Operator Plane)



Consulting Pythagoras and the algebra in Endnote VIII, the magnitude of the resultant vector $|\underline{V}_3|$ from Figure 13 would be:

$$|\underline{V}_3| = \sqrt{[(\underline{A}*\underline{B} - \underline{a}*b)^2 + ((\underline{A}*\underline{b} + \underline{B}*\underline{a}))^2]}$$

Would that match the Polar Shortcut formula where $|\underline{V}_3| = |\underline{V}_1| * |\underline{V}_2|$??

$$|\underline{V}_1| = \sqrt{[A^2 + a^2]}$$

$$|\underline{V}_2| = \sqrt{[B^2 + b^2]}$$

Does? $\sqrt{[(\underline{A}*\underline{B} - \underline{a}*b)^2 + ((\underline{A}*\underline{b} + \underline{B}*\underline{a}))^2]}$?=? $(\sqrt{[A^2 + a^2]}) * (\sqrt{[B^2 + b^2]})$

Multiply them out each side (for the Algebra, see end note) and find that each side gets to:

$$\sqrt{[A^2 B^2 + a^2 b^2 + A^2 b^2 + B^2 a^2]}$$

YES! The magnitudes of the resultants do match, the polar short cut on magnitude is justified!

[x] **Endnote X: Checking out the Argument (Angle) Short Cut:**

How about the angle (or "argument") of the resultant vector $\underline{V}_3$?

The algebraic solution from Figure 13 (repeated in Endnote IX) suggests that:

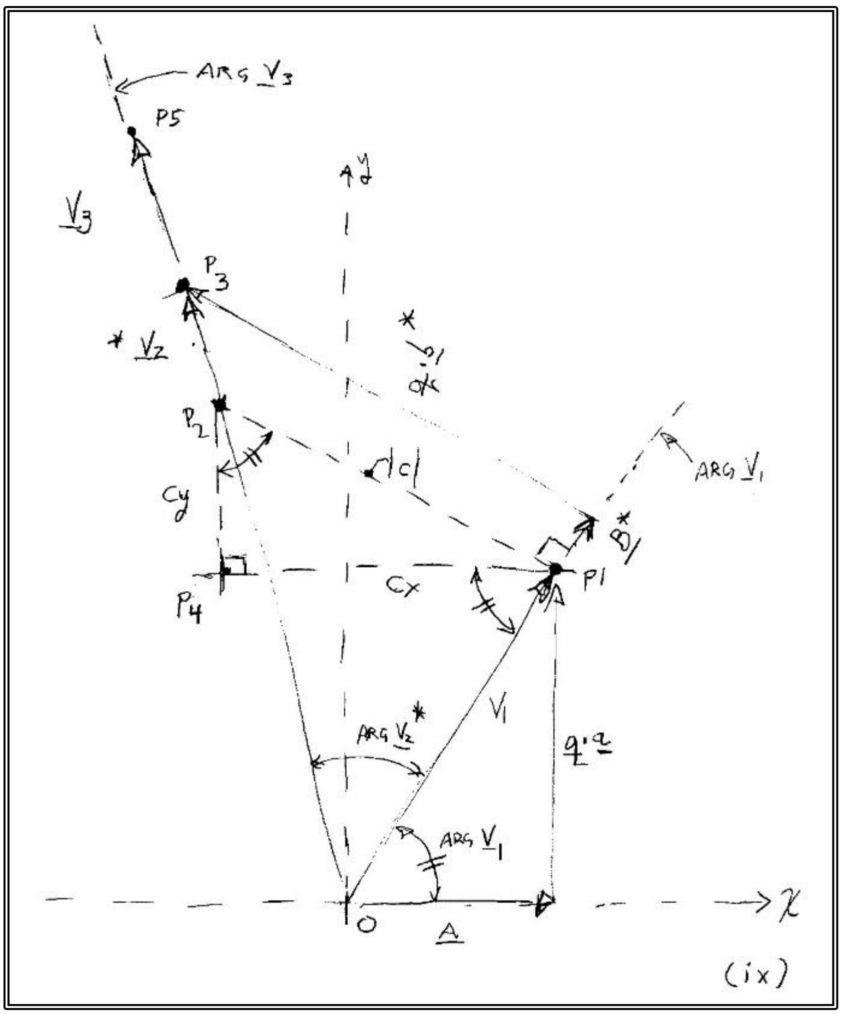
ArgV3 = atan [(Ab+Ba)/(AB-ab)]

Would this match the polar shortcut where Arg $\underline{V}_3$ = Arg $\underline{V}_1$ + Arg $\underline{V}_2$. ??

Here we resort to a graphical approach shown in Figure 14: Vector *$\underline{V}_2$ is vector $\underline{V}_2$ from Figure13 re-drawn as rotated (*) by Arg $\underline{V}_1$ and placed tangent to vector $\underline{V}_1$ so that their arguments add together as shown. Note that in this figure *$\underline{B}$ is superimposed over $\underline{V}_1$, and the resultant $\underline{V}_3$ is superimposed over *$\underline{V}_2$ This construction reflects the "logic" of the polar short cut.

The proof strategy will be to show that the polar angle through the construction point P2 will match the argument Arg $\underline{V}_3$ from the algebraic solution. P2 is the intersection of $\underline{V}_2$ and a line perpendicular to $\underline{V}_1$ through point P1. (To be clear, P1 is the point at the end of vector $\underline{V}_1$.). Then, since P3 and P5 both lie along this same angle through P2, we will see that the algebraic solution and the polar short cut arguments match.

FIGURE 14 (Repeated) ($\underline{q}$ Operator Plane)



Since triangle (**0-P1-P2**) is similar to triangle (**0-B-P3**) we can find |**C**|:

$$|\mathbf{C}| = |\underline{\mathbf{V}}_1| * (\mathbf{b}/\mathbf{B})$$

Euclid would easily show us that the angles marked with double slash (//) are all equal to Arg V1. This means that the triangle with sides **C, Cx** and **Cy** is similar to the triangle with sides **V1, A**, and **a**. Therefore, we can find **Cx** and **Cy**.

$$\mathbf{Cx} = |\mathbf{C}| *(\mathbf{a} / |\mathbf{V}_1|) = |\underline{\mathbf{V}}_1| * (\mathbf{b}/\mathbf{B}) * (\mathbf{a} / |\mathbf{V}_1|) = (\mathbf{a} * \mathbf{b})/\mathbf{B}$$

$$\mathbf{Cy} = |\mathbf{C}| *(\mathbf{A} / |\mathbf{V}_1|) = |\underline{\mathbf{V}}_1| * (\mathbf{b}/\mathbf{B}) * (\mathbf{A} / |\mathbf{V}_1|) = (\mathbf{b}*\mathbf{A})/\mathbf{B}$$

Then point P2 can be found by adding sides (**A – Cx**) and ***q****(**a + Cy**). From this we can find the argument for the vector through point P2:

$$\text{Arg } (\underline{\mathbf{P2}}) = \text{atan } (\mathbf{a}+ \mathbf{Cy})/(\mathbf{A} - \mathbf{Cx}).$$

$$\text{Arg } (\underline{\mathbf{P2}}) = \text{atan } [(\mathbf{a} + (\mathbf{b}*\mathbf{A})/\mathbf{B}) / (\mathbf{A} - (\mathbf{a}*\mathbf{b})/\mathbf{B})]$$

We can simplify the [ ] quantity , just multiply by (**B**/**B**)

$$\text{Arg } (\underline{\mathbf{P2}}) = \text{atan } [((\mathbf{a}*\mathbf{B}) + (\mathbf{A}*\mathbf{b})) / ((\mathbf{B}*\mathbf{A}) - (\mathbf{a}*\mathbf{b}))]$$

This can be rewritten:

$$\text{Arg } (\underline{\mathbf{P2}}) = \text{atan } [(\mathbf{Ab}+\mathbf{Ba})/(\mathbf{AB}-\mathbf{ab})]$$

This is the angle through point P2 in the logic of the Polar "Shortcut." As shown in figure 14, this is also the angle through points P3 and P5. Therefore this is the Arg($\underline{V}_3$), namely the resultant vector from the Polar Shortcut calculation. It matches the algebraic solution of Figure 13:

$$\text{Arg}(\underline{\mathbf{V}}_3) = \text{atan } [(\mathbf{Ab}+\mathbf{Ba})/(\mathbf{AB}-\mathbf{ab})]$$

So, the algebraic solution for the Argument of **V3** matches the geometric angle of adding the arguments of **V1** and **V2**.

$$\text{Arg } (\underline{\mathbf{P2}}) = \text{Arg } (\underline{V}_3) = \text{Arg}(\underline{V}_1) + \text{Arg}(\underline{V}_2)$$

"Q.E.D." *(.... always wanted a chance to say that!)*

YES! The polar shortcut for finding the product argument by simply adding the component arguments is valid.

[xi] **Endnote XI: Mapping (~>) vs. equality (=)**

The notation (~>) is used here to indicate a 1:1 mapping onto the unit circle. It is unclear how much this is a true "equivalence" and if the mapping is truly bi-directional. Despite the coincidental mapping of each function onto the unit circle, are we ready to make the leap that

$$(-1)^{x/\pi} = e^{ix} \quad ????$$

Let us try to stick with the notation that:

$$(-1)^{x/\pi} \sim> e^{ix}$$

[xii] **<u>Endnote XII: The UNIT CIRCLE and scaling:</u>**

The unit circle is very useful for conceptual work, so the physical constants of mass, spring force, and ΔL were chosen to keep the answers on the unit circle. Obviously, the physical response would be remarkably similar if extra energy was added to the system, but the numerical responses might not fit the unit circle. This is easily adjusted by "normalizing" the response to a proportion of the total energy. Likewise, the choice of units (feet, vs meters, etc.) might also necessitate other scaling. Again, a choice of perspective can make a circle appear as an ellipse, and vice versa. This adventure skirts this refinement in the interest of intuition.

[xiii] **<u>ENDNOTE XIII: Alternate Approach:</u>**

While editing this draft I came across this YouTube video by 3Blue1Brown that does a wonderful job of covering some of this same territory by a different path. Well worth watching, especially once warmed up by the ***Fool's Errand Adventure***.

https://www.youtube.com/watch?v=-j8PzkZ70Lg

[xiv] **<u>ENDNOTE XIV : Title Page:</u>**

The Moon Snail Volute sketched for the title page cries out for exponential spirals. Just re-imagine the cylindrical 3-dimensional graph of Fig 20 being changed into a spherical-coordinate-system, (R,Θ,Φ) where angle Φ replaces the linear time axis. It would be like an exponential spiral on a rotisserie....

To approximate the shape of the moon snail shell we can extend the polar equation into spherical 3-dimensional plot where both sphere angles are growing linearly with time: $\Theta = \omega * t$, and $\Phi = t$.

$$\underline{\boldsymbol{R}}\{\Theta,\Phi\} = \underline{\boldsymbol{R}}\{\Theta,t\} = \underline{\boldsymbol{R}}\{(\omega * t), t\} = (e^{q\omega t})*(e^{at}) \quad \text{eq 27}$$

Here ($e^{q\omega t}$) is the underlying circular motion and sinusoidal oscillation of Figure 19 and (e^{at}) is the slowly <u>expanding</u> exponential. This equation 27 looks remarkably similar to eq 26 of figure 20 (changing only **"-a"** for decay into **"+a"** for growth). However, this small change, together with the shift to the spherical perspective, winds the spiral into a <u>growing</u> volute. The actual moon snail shell has some skew and twist not included in eq 27, but still ......